What Is a Single Photon and How to Detect It?

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The notion of a photon was first proposed by Einstein in 1905, his year of miracles, in an attempt to explain the photoelectric effects. The subsequent experiment conducted by Compton demonstrated that the photon also carries momentum, which gave a strong evidence of the existence of a single photon. However, the satisfactory description incorporating both wave and particle nature of light didn't appear until the advent of quantum mechanics. In this paper, we will use the quantum mechanical description of light to explore the question: what is a single photon? We start by reviewing the quantization of a single mode of electromagnetic field and the photon state introduced in 8.05x. We will move on to extend our discussion to the quantization of a general electromagnetic field. One-photon multimode state is then introduced, which is the state analogous to an isolated corpuscle of light traveling in spacetime with speed of light. To answer the question that where the photon is, the principle of photodetectors is introduced. The detection probability of a photon at a given spacetime point is derived. We conclude with a phenomenon unique to a single photon: it can not be detected at different places at the same time!

I. INTRODUCTION

Before the 20th century, light was described successfully by Maxwell's equations and was thought to be electromagnetic waves. In 1905, Einstein treated light as photons to explain photoelectric effect successfully, while Maxwell's theory of light had difficult to explain such phenomenon. It was not clear why light sometimes behaves like waves but sometimes behaves like particles. The unified theory of light is the quantum version of Maxwell's theory, also known as quantum electrodynamics. The notion of a single photon is best understood under the framework of quantum electrodynamics.

The particle nature of light is usually demonstrated using light with extremely low intensity. In double slits experiments, in order to rule out the possibility that interaction among photons causes the interference pattern, light source with extremely low intensity has been used and it is said that photons appear on the screen one by one. Feynman explained the particle nature of light in his lecture [1], arguing that the light always comes as lump when we detect it. However, it should be emphasized that these arguments are inappropriate. Low intensity only indicates the average of photon number is much smaller than one, but not necessarily means there is only one photon in spacetime. In addition, the discrete clicks we hear in the detector can be well explained by the quantum nature of the detector alone, while the quantization of light is not indispensable.

This paper focuses on giving a unified description of a single photon. In section II, we start with the wave description of light, Maxwell's equations, and then construct its quantum version. In section III, we will see the notion of photons comes looking for us when we try to find the eigenstates of Hamiltonian of the electromagnetic field. Single-photon multimode state is introduced, which is the quantum state of light analogous to an isolated photon in spacetime. Finally, in section IV, the principle of photodetectors is introduced, and we calculate the detection probability of a single photon at a given spacetime point. A phenomenon unique to a single photon that makes it different from classical state of light is that the double detection probability at different places at the same time is zero.

II. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

In chapter 9 of 8.05x, photon states are introduced, where we examined a single mode of the electromagnetic field in a rectangular cavity with frequency ω and wavenumber $k = \omega/c$, with c the speed of light in vacuum. We learned that the Hamiltonian of a single mode electromagnetic field resembles that of a one-dimensional harmonic oscillator with electric field acting like position variable and magnetic field acting like momentum variable (which is which is more of a convention). After promoting both dynamic variables to operators and imposing the canonical commutation relation, we construct the quantum theory of a single mode of the electromagnetic field. The classical electric and magnetic fields both become field operators.

We extend our discussion to an electromagnetic field with many modes. Since we have seen this topic in class using a specific example of the electromagnetic field in a rectangular cavity, we will make our discussion more general here. We will find that, due to the orthogonality between different modes, the extended version Hamiltonian is just the sum of different single mode Hamiltonians. Various vector identities used in this section can be found in Jackson's book [2] or be conveniently proved using index notation to write everything in its component form and applying the identity about Levi-Civita symbol $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$. The discussion here follows Ballentine [3] and Aspect [4] closely.

The energy of the electromagnetic field is

$$E_{\mathrm{EM}} = (8\pi)^{-1} \int d^3x \left(\mathbf{E}^2(\mathbf{x}, t) + \mathbf{B}^2(\mathbf{x}, t) \right). \quad (1)$$

Notice Gaussian unit is used in this paper. We will see in the end of this section that Eq. (1) can be put into a nicer form that is convenient for us to postulate its quantum version. In vacuum, with no electric charge, $\mathbf{E}$ and $\mathbf{B}$ satisfies source-free Maxwell's equations

$$\nabla \cdot \mathbf{E} = 0, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{5}$$

which are called Gauss's law, no magnetic monopole, Faraday's law and Ampère's law respectively. We can eliminate $\mathbf{B}$ in Maxwell's equations to get one second order differential equation for $\mathbf{E}$: take the time derivative of Ampère's law Eq. (5), use Faraday's law Eq. (4) to eliminate $\mathbf{B}$, apply the vector identity $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$, and use Gauss's law Eq. (2) to have

$$\frac{1}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = -c\nabla \times (\nabla \times \mathbf{E}) = c\nabla^2 \mathbf{E}. \tag{6}$$

Apply our familiar separation of variable technique to write $\mathbf{E}(\mathbf{x}, t) = 2\sqrt{\pi}\omega q(t)\mathbf{u}(\mathbf{x})$, where the strange prefactor $2\sqrt{\pi}\omega$ is put there to make the final expression of energy looks nicer. Plug this ansatz into Eq. (6) and make some rearrangements

$$\nabla^2 \mathbf{u}(\mathbf{x}) = \left(\frac{1}{c^2}\frac{d^2 q(t)}{dt^2}\frac{1}{q(t)}\right)\mathbf{u}(\mathbf{x}). \tag{7}$$

It can be seen that $(d^2q/dt^2)(1/q)$ should be a constant, and we let it to be $-\omega^2$, or otherwise $\mathbf{u}(\mathbf{x})$ would have time dependence. Then Eq. (7) gives

$$\frac{d^2 q}{dt^2} = -\omega^2 q, \tag{8}$$

$$\nabla^2 \mathbf{u}(\mathbf{x}) = -\left(\frac{\omega}{c}\right)^2 \mathbf{u}(\mathbf{x}). \tag{9}$$

The spatial part Eq. (9) has a similar structure to the time-independent Schrödinger equation. After applying some boundary conditions, we will get a complete set of eigenfunctions $\mathbf{u}_m(\mathbf{x})$ and eigenvalues ω_m labeled by integer m. For each mode with eigenvalue ω_m, Eq. (8) can be solved for $q_m(t)$. $\mathbf{u}_m(\mathbf{x})$ is sometimes called mode function and can be chosen to satisfy the orthonormality condition

$$\int d^3x \mathbf{u}_{m'}(\mathbf{x}) \cdot \mathbf{u}_m(\mathbf{x}) = \delta_{m'm}. \tag{10}$$

A general solution is the linear combination of solutions with different m

$$\mathbf{E}(\mathbf{x}, t) = \sum_m 2\sqrt{\pi}\omega_m q_m(t)\mathbf{u}_m(\mathbf{x}), \tag{11}$$

and $\mathbf{B}$ can be obtained by plugging Eq. (11) into Faraday's law Eq. (4)

$$\mathbf{B}(\mathbf{x}, t) = \sum_m 2\sqrt{\pi}\frac{c}{\omega_m} p_m(t)\nabla \times \mathbf{u}_m(\mathbf{x}), \tag{12}$$

with $dp_m/dt = -\omega_m^2 q_m$. The curl of mode functions also has orthogonality property that the integral $\int d^3x(\nabla \times \mathbf{u}_{m'}(\mathbf{x})) \cdot (\nabla \times \mathbf{u}_m(\mathbf{x}))$ vanishes if $m' \neq m$. To show this, apply another vector identity $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ with $\mathbf{a} = \mathbf{u}_{m'}(\mathbf{x})$ and $\mathbf{b} = \nabla \times \mathbf{u}_m(\mathbf{x})$

$$\begin{aligned}\nabla \cdot [\mathbf{u}_{m'}(\mathbf{x}) \times (\nabla \times \mathbf{u}_m(\mathbf{x}))] &= (\nabla \times \mathbf{u}_{m'}(\mathbf{x})) \cdot (\nabla \times \mathbf{u}_m(\mathbf{x})) \\ &- \mathbf{u}_{m'}(\mathbf{x}) \cdot (\nabla \times (\nabla \times \mathbf{u}_m(\mathbf{x}))).\end{aligned} \tag{13}$$

The second term in Eq. (13) can be simplified using the same trick in Eq. (6) as we eliminate $\mathbf{B}$ in Maxwell's equations

$$\begin{aligned}\mathbf{u}_{m'}(\mathbf{x}) \cdot (\nabla \times (\nabla \times \mathbf{u}_m(\mathbf{x}))) &= -\mathbf{u}_{m'}(\mathbf{x}) \cdot \nabla^2 \mathbf{u}_m(\mathbf{x}) \\ &= \left(\frac{\omega_m}{c}\right)^2 \mathbf{u}_{m'}(\mathbf{x}) \cdot \mathbf{u}_m(\mathbf{x}),\end{aligned} \tag{14}$$

where in the last step, we use the differential equation for $\mathbf{u}$ in Eq. (9). Integrate Eq. (13) over the space to have

$$\begin{aligned}&\int d^3x(\nabla \times \mathbf{u}_{m'}(\mathbf{x})) \cdot (\nabla \times \mathbf{u}_m(\mathbf{x})) \\ &= \left(\frac{\omega_m}{c}\right)^2 \delta_{m'm} + \int d^3x \nabla \cdot [\mathbf{u}_{m'}(\mathbf{x}) \times (\nabla \times \mathbf{u}_m(\mathbf{x}))] \\ &= \left(\frac{\omega_m}{c}\right)^2 \delta_{m'm}.\end{aligned} \tag{15}$$

The volume integral in second line of Eq. (15) can be shown to vanish by converting it to surface integral on the boundary and use the fact that $\mathbf{u}_{m'}(\mathbf{x})$ is perpendicular to the conducting surface. We are now ready to calcualte the energy of electromagnetic field. Let us plug $\mathbf{E}$ in Eq. (11) and $\mathbf{B}$ in Eq. (12) into the energy of electromagnetic field in Eq. (1)

$$\begin{aligned}E_{\text{EM}} &= \frac{1}{2}\sum_{m',m}\int d^3x \big[\omega_{m'}\omega_m q_{m'}(t) q_m(t)\mathbf{u}_{m'}(\mathbf{x}) \cdot \mathbf{u}_m(\mathbf{x}) \\ &+ \frac{c}{\omega_{m'}}\frac{c}{\omega_m} p_{m'}(t) p_m(t)(\nabla \times \mathbf{u}_{m'}(\mathbf{x})) \cdot (\nabla \times \mathbf{u}_m(\mathbf{x}))\big] \\ &= \frac{1}{2}\sum_m \left[\omega_m^2 q_m^2(t) + p_m^2(t)\right],\end{aligned} \tag{16}$$

where we use the orthonormality condition for $\mathbf{u}_m$ in Eq. (10) and the orthogonality condition for $\nabla \times \mathbf{u}_m$ in Eq. (15) in the last step.

Equation (16) is our desired result. Compared with what we learned in 8.05x, where the energy of a single mode of the electromagnetic field is $E = 1/2\left(p^2(t) + \omega^2 q^2(t)\right)$, we notice the energy of multimode electromagnetic field is simply the sum of the energy of different modes. Let us go quantum! The procedure is similar to what we did in class. The only difference is that we have a summation symbol here. We postulate a Hamiltonian by promoting dynamic variables p_m and q_m to operators

$$\hat{H}_{\rm EM} = \frac{1}{2}\sum_m \left(\hat{p}_m^2 + \omega_m^2 \hat{q}_m^2\right), \tag{17}$$

where $\hat{p}_m$ and $\hat{q}_m$ are the promoted Schrödinger operators and they satisfy the canonical commutation relation $[\hat{q}_{m'}, \hat{p}_m] = i\hbar\delta_{m'm}$. The Hamiltonian is the same as a system of independent harmonic oscillators. As usual, we can define annihilation and creation operators associated with m-th mode, $\hat{a}_m$ and $\hat{a}_m^\dagger$, as

$$\hat{a}_m = \frac{1}{\sqrt{2\hbar\omega_m}}(\omega_m \hat{q}_m + i\hat{p}_m), \tag{18}$$

$$\hat{a}_m^\dagger = \frac{1}{\sqrt{2\hbar\omega_m}}(\omega_m \hat{q}_m - i\hat{p}_m). \tag{19}$$

It can be shown immediately that the commutation relation in terms of $\hat{a}_m$ and $\hat{a}_m^\dagger$ is

$$[\hat{a}_{m'}, \hat{a}_m^\dagger] = \delta_{m'm}, \tag{20}$$

and the Hamiltonian can be written using creation and annihilation operators

$$\begin{aligned}\hat{H}_{\rm EM} &= \sum_m \hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}\right) \\ &= \sum_m \hbar\omega_m \left(\hat{N}_m + \frac{1}{2}\right), \end{aligned}\tag{21}$$

where the number operator is defined as usual, $\hat{N}_m = \hat{a}_m^\dagger \hat{a}_m$. The electric field $\mathbf{E}$ becomes a field operator after we promote p_m and q_m to operators

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{x}) &= \sum_m 2\sqrt{\pi}\omega_m \hat{q}_m \mathbf{u}_m(\mathbf{x}) \\ &= \sum_m \sqrt{2\pi\hbar\omega_m}(\hat{a}_m + \hat{a}_m^\dagger)\mathbf{u}_m(\mathbf{x}). \end{aligned}\tag{22}$$

It should be emphasized that the $\mathbf{x}$ in Eq. (22) is a label to indicate which operator we are talking about. It should not be confused as position operators $\hat{\mathbf{x}}$, which is a dynamic variable of a particle, or an observable.

III. NOTION OF A SINGLE PHOTON

Now we have the Hamiltonian of electromagnetic field in hand, it is time to find its spectrum. It is an easy task for us since the Hamiltonian is the same as a collection of independent harmonic oscillators. The ground state $|0\rangle$ is the state that is killed by all annihilation operators $\hat{a}_m$

$$\hat{a}_m |0\rangle = 0, \forall m. \tag{23}$$

The ground state is labeled by number 0 because it is eigenstate for all number operators $\hat{N}_m = \hat{a}_m^\dagger \hat{a}_m$ with eigenvalue 0, $\hat{N}_m |0\rangle = \hat{a}_m^\dagger \hat{a}_m |0\rangle = 0, \forall m$. If we define total number operator

$$\hat{N} = \sum_m \hat{N}_m, \tag{24}$$

then the state $|0\rangle$ is also its eignestate with eigenvalue 0. For this reason, the ground state is also known as vacuum, because it represents a world with nothing in it. A general energy eigenstate is built by acting creation operators repeatedly on vacuum and is labeled by various eigenvalues of $\hat{N}_m$

$$|n_1, n_2, ..., n_m, ...\rangle = (\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2} ... (\hat{a}_m^\dagger)^{n_m} ... |0\rangle, \tag{25}$$

where the order of creation operators in right-hand side is not important since they all commute. The state in Eq. (25) is interpreted to have $n_1, n_2, ..., n_m, ...$ photons with frequency $\omega_1, \omega_2, ..., \omega_m, ...$ respectively.

A. One-photon Multimode State

Now it is natural to introduce the state that is analogous to an isolated corpuscle of light. What properties of the state do we expect if it represents a single photon propagating in spacetime at the speed of light? A single photon can have any frequency it likes but one thing we know for sure is that the total photon number should be one in such a state. Can we construct eigenstates of total number operator $\hat{N}$ with eigenvalue 1? The answer is yes and it is not hard. The state $\hat{a}_m^\dagger |0\rangle$ represents one photon with frequency ω_m, and it is an eigenstate of $\hat{N}$ with eigenvalue 1. The linear combination of such states with all possible m values will also be eigenstates of $\hat{N}$ with eigenvalue 1. Let us define

$$|1\rangle = \sum_m c_m \hat{a}_m^\dagger |0\rangle, \tag{26}$$

with c_m some coefficients. The state with the form given in Eq. (26) is called *one-photon multimode state*. We can check that

$$\hat{N}|1\rangle = \sum_m c_m \hat{N}\hat{a}_m^\dagger |0\rangle = \sum_m c_m \hat{a}_m^\dagger |0\rangle = |1\rangle. \tag{27}$$

To make the state well-normalized, the coefficients should satisfy $\sum_m |c_m|^2 = 1$. Since the number operator $\hat{N}_m$ commutes with the Hamiltonian $\hat{H}_{\rm EM}$, the total number operator $\hat{N}$ commutes with $\hat{H}_{\rm EM}$ too, and the total photon number is a conserved quantity. If the initial state of electromagnetic field is a one-photon multimode state of the form in Eq. (26) with some known coefficients, the time-evolved state $|1(t)\rangle$ will still be a one-photon multimode state. It looks we are on the right track.

B. Where is Our Photon?

Next, let us explore whether it is possible to recover the classical picture of a single photon flying in spacetime with speed of light. The first guess is that we can calculate the expectation value of the electric field operator. Consider a single photon that is localized at some region at a given time t, our hope is that the expectation value of the electric field operator in this region should be larger. However, this is not true, as will be shown below. In order to calculate the expectation value of the electric field operator at time t, we choose to use Heisenberg's picture here. The Heisenberg operator version of $\hat{\mathbf{E}}(\mathbf{x})$ in Eq. (22) can be obtained by replacing creation and annihilation operators on the right-hand side with their Heisenberg operator version

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{x},t) &= \sum_m 2\sqrt{\pi}\omega_m \hat{q}_m(t)\mathbf{u}_m(\mathbf{x}) \\ &= \sum_m \sqrt{2\pi\hbar\omega_m}\left(\hat{a}_m(t)+\hat{a}_m^\dagger(t)\right)\mathbf{u}_m(\mathbf{x}),\end{aligned} \tag{28}$$

where we add the time dependence to indicate Heisenberg operator. We have calculated $\hat{a}_m(t)$ and $\hat{a}_m^\dagger(t)$ several times in class. As a review, we repeat the calculation here. There is no explicit time dependence in $\hat{H}_{\mathrm{EM}}$ so the time evolution operator is $\hat{U}(t) = \exp(-it\hat{H}_{\mathrm{EM}}/\hbar)$. By definition, Heisenberg operator version of annihilation operator $\hat{a}_m$ is $\hat{a}_m(t) = \exp(it\hat{H}_{\mathrm{EM}}/\hbar)\hat{a}_m\exp(-it\hat{H}_{\mathrm{EM}}/\hbar) = \exp(i\omega_m t\hat{N}_m)\hat{a}_m\exp(-i\omega_m t\hat{N}_m)$. In the second step, we use the fact that all terms with subscript not equal to m in $\hat{H}_{\mathrm{EM}}$ commute through $\hat{a}_m$.We then take the time derivative to get a differential equation $\dot{\hat{a}}_m(t) = i\omega_m\exp(i\omega_m t\hat{N}_m)[\hat{N}_m,\hat{a}_m]\exp(-i\omega_m t\hat{N}_m) = -i\omega_m\hat{a}_m(t)$. The initial condition is $\hat{a}_m(t=0) = \hat{a}_m$ so we have $\hat{a}_m(t) = \exp(-i\omega_m t)\hat{a}_m$. Take the hermitian conjugate to get $\hat{a}_m^\dagger(t) = \exp(i\omega_m t)\hat{a}_m^\dagger$. Plug this results into Heisenberg operator $\hat{\mathbf{E}}(\mathbf{x},t)$ in Eq. (28),

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{x},t) &= \sum_m \sqrt{2\pi\hbar\omega_m}\left(\hat{a}_m e^{-i\omega_m t}+\hat{a}_m^\dagger e^{i\omega_m t}\right)\mathbf{u}_m(\mathbf{x}) \\ &= \hat{\mathbf{E}}^+(\mathbf{x},t)+\hat{\mathbf{E}}^-(\mathbf{x},t),\end{aligned} \tag{29}$$

where in the second step we break the electric field operator into positive and negative parts with the definitions

$$\hat{\mathbf{E}}^+(\mathbf{x},t) = \sum_m \sqrt{2\pi\hbar\omega_m}\hat{a}_m e^{-i\omega_m t}\mathbf{u}_m(\mathbf{x}), \tag{30}$$

$$\hat{\mathbf{E}}^-(\mathbf{x},t) = \sum_m \sqrt{2\pi\hbar\omega_m}\hat{a}_m^\dagger e^{i\omega_m t}\mathbf{u}_m(\mathbf{x}). \tag{31}$$

The reason for this decomposition will become clear in the next section where we discuss detection of photons. For now just remember there are two parts in the electric field operator, positive part containing all annihilation operators and negative part containing all creation operators. The expectation value of the electric field operator is

$$\begin{aligned}\langle 1|\,\hat{\mathbf{E}}(\mathbf{x},t)\,|1\rangle = \sum_{m,k,l}\sqrt{2\pi\hbar\omega_m}\big(c_l^* c_k\,\langle 0|\,\hat{a}_l\hat{a}_m\hat{a}_k^\dagger\,|0\rangle\, e^{-i\omega_m t} \\ + c_l^* c_k\,\langle 0|\,\hat{a}_l\hat{a}_m^\dagger\hat{a}_k^\dagger\,|0\rangle\, e^{i\omega_m t}\big)\mathbf{u}_m(\mathbf{x}) = 0.\end{aligned} \tag{32}$$

To see why it vanishes, let us look closely at the factor $\langle 0|\,\hat{a}_l\hat{a}_m\hat{a}_k^\dagger\,|0\rangle$. There is one creation operator but two annihilation operators, so by no means can we have $\hat{a}_l\hat{a}_m\hat{a}_k^\dagger\,|0\rangle \propto |0\rangle$, as a result $\langle 0|\,\hat{a}_l\hat{a}_m\hat{a}_k^\dagger\,|0\rangle = 0$. Same reason works for the other factor with two creation operators but only one annihilation operator. Nothing interesting happens if we only look at the expectation value of the electric field operator. They average to zero in one-photon multimode state $|1\rangle$.

However, what about the mean square of the electric field operator, $\langle 1|\,\hat{\mathbf{E}}(\mathbf{x},t)\cdot\hat{\mathbf{E}}(\mathbf{x},t)\,|1\rangle$? We know the mean square of the electric field is proportional to the intensity of light in classical electromagnetic theory so it is reasonable to expect this quantity peaks in the region where the single photon is localized. Let us calculate! The square of $\hat{\mathbf{E}}(\mathbf{x},t)$ is

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{x},t)\cdot\hat{\mathbf{E}}(\mathbf{x},t) = \hat{\mathbf{E}}^+(\mathbf{x},t)\cdot\hat{\mathbf{E}}^+(\mathbf{x},t)+\hat{\mathbf{E}}^-(\mathbf{x},t)\cdot\hat{\mathbf{E}}^-(\mathbf{x},t) \\ +\hat{\mathbf{E}}^-(\mathbf{x},t)\cdot\hat{\mathbf{E}}^+(\mathbf{x},t)+\hat{\mathbf{E}}^+(\mathbf{x},t)\cdot\hat{\mathbf{E}}^-(\mathbf{x},t).\end{aligned} \tag{33}$$

We calculate term by term. Moments of first two terms vanish: $\langle 1|\,\hat{\mathbf{E}}^+(\mathbf{x},t)\cdot\hat{\mathbf{E}}^+(\mathbf{x},t)\,|1\rangle = \langle 1|\,\hat{\mathbf{E}}^-(\mathbf{x},t)\cdot\hat{\mathbf{E}}^-(\mathbf{x},t)\,|1\rangle = 0$ because the number of creation operators is not equal to that of annihilation operators between the vacuum $\langle 0|\ldots|0\rangle$. For the third term in Eq. (33)

$$\begin{aligned}\hat{\mathbf{E}}^-(\mathbf{x},t)\cdot\hat{\mathbf{E}}^+(\mathbf{x},t) = \sum_{m,n}2\pi\hbar\sqrt{\omega_m\omega_n} \\ \times(\mathbf{u}_m\cdot\mathbf{u}_n)\hat{a}_m^\dagger\hat{a}_n e^{i(\omega_m-\omega_n)t},\end{aligned} \tag{34}$$

while we have

$$\begin{aligned}\langle 1|\,\hat{a}_m^\dagger\hat{a}_n\,|1\rangle &= \sum_{k,l}c_k^* c_l\,\langle 0|\,\hat{a}_k\hat{a}_m^\dagger\hat{a}_n\hat{a}_l^\dagger\,|0\rangle \\ &= \sum_{k,l}c_k^* c_l\,\langle 0|\,[\hat{a}_k,\hat{a}_m^\dagger][\hat{a}_n,\hat{a}_l^\dagger]\,|0\rangle \\ &= \sum_{k,l}c_k^* c_l\delta_{km}\delta_{nl}\,\langle 0|0\rangle = c_m^* c_n.\end{aligned} \tag{35}$$

The moment of the third term in Eq. (33) is thus

$$\begin{aligned}\langle 1|\,\hat{\mathbf{E}}^-(\mathbf{x},t)\cdot\hat{\mathbf{E}}^+(\mathbf{x},t)\,|1\rangle = \sum_{m,n}2\pi\hbar\sqrt{\omega_m\omega_n} \\ \times(\mathbf{u}_m\cdot\mathbf{u}_n)c_m^* c_n e^{i(\omega_m-\omega_n)t}.\end{aligned} \tag{36}$$

The last term in Eq. (33) can be calculated in the same manner. We state the result here

$$\begin{aligned}\langle 1|\,\hat{\mathbf{E}}^+(\mathbf{x},t)\cdot\hat{\mathbf{E}}^-(\mathbf{x},t)\,|1\rangle = \sum_{m,n}2\pi\hbar\sqrt{\omega_m\omega_n} \\ \times(\mathbf{u}_m\cdot\mathbf{u}_n)c_n^* c_m e^{i(\omega_n-\omega_m)t}+\sum_m 2\pi\hbar\omega_m\mathbf{u}_m^2.\end{aligned} \tag{37}$$

Combine the results in Eq. (36) and Eq. (37) to get the mean square of the electric field operator

$$\begin{aligned}\langle 1|\,\hat{\mathbf{E}}(\mathbf{x},t)\cdot\hat{\mathbf{E}}(\mathbf{x},t)\,|1\rangle = \sum_m 2\pi\hbar\omega_m\mathbf{u}_m^2+\sum_{m,n}2\pi\hbar\sqrt{\omega_m\omega_n} \\ \times(\mathbf{u}_m\cdot\mathbf{u}_n)2\Re(c_m^* c_n e^{i(\omega_m-\omega_n)t}),\end{aligned} \tag{38}$$

where $\Re(z)$ means real part of a complex number z. The first term in Eq. (38) does not have c_m dependence or time dependence but only depends on the spectrum of the configuration. The second term can describe the disturbance propagating in spacetime. The peak of the distribution in space at time t is the region where the single photon is localized. We will give our statement here a physical picture in the next section, where we will find the moment of $\hat{\mathbf{E}}^{-}(\mathbf{x},t)\cdot\hat{\mathbf{E}}^{+}(\mathbf{x},t)$ in Eq. (36) will affect detection rate of a photodetector placed at position $\mathbf{x}$ at time t.

Still, we want to answer the question: where is the photon, more precisely. Is it possible to define a position operator $\hat{\mathbf{X}}$ similar to what we have in non-relativistic quantum mechanics for a single particle, so the eigenstate of this position operator $|\mathbf{x}\rangle$ represents a photon at position $\mathbf{x}$? Sidney Coleman argued in one of his quantum field theory lectures [5] that although such position operator can be well-defined, it will lead to unphysical results like traveling faster than the speed of light in vacuum. He argued further that the underlying physical reason is that once you try to localize a single particle to a small enough region, the uncertainty in its momentum will become so large that pair production will occur. As a result, we do not know whether we still have a single particle. The field operators are the right tool to use since they deal with this relativity causality automatically. In classical electrodynamics, the information of the existence of a charged particle in vacuum will travel at speed of light outwards in form of electromagnetic radiation. Outside the wavefront, the field is zero. In order to get an answer to the question: where is the photon, we should rephrase the question and ask "what is the probability to detect the photon at spacetime point $(\mathbf{x},t)$?", which is a more appropriate question in quantum mechanics. This leads us to our discussion on photon detection.

IV. DETECTION OF A SINGLE PHOTON

We claimed in previous section that the mean square of the electric field operator in Eq. (38) can be thought as intensity of the light, and the peak of this quantity in space is the region where the single photon localized. In this section, we will develop this idea and go deeper. First, let us think what the intensity of the light means when we have a single photon. Classically, when the intensity of light is larger, it means the light is brighter. When the brighter light hits on the retina in our eyes, more excited our optic nerve will be, which means the detection rate of our eye becomes larger. If we have a single photon, intensity of the light at spacetime point $(\mathbf{x},t)$ is naturally related to the probability of detecting the photon at position $\mathbf{x}$ and time t.

We have apparatus called photodetector to detect photons through its interaction with light. When light shines on such apparatus, the photons will kick a bounded electron in the atom out into its continuous spectrum. The signal of an outgoing electron is amplified, and then we can hear a click. We have encountered such process in 8.06x when we talked about ionization and light-atom interaction in chapter 5 and 6, where we focused on quantization of atom and treated the electric field as a classical object. The effect of electric field with amplitude E_0 enters the transition rate as a factor E_0^2. Similar results will be found here when we treat the electric field as a quantum mechanical object. The discussion in this section follows Ballentine [3], Aspect [4] and Cohen-Tannoudji [6] closely.

A. Principle of Photodetectors

The Hamiltonian of an atom-light system $\hat{H}$ has three pieces: atom contribution $\hat{H}_{\mathrm{at}}$, electromagnetic field contribution $\hat{H}_{\mathrm{EM}}$ and interaction of atom and field contribution $\delta\hat{H}$

$$\hat{H}=\hat{H}_{\mathrm{at}}+\hat{H}_{\mathrm{EM}}+\delta\hat{H}=\hat{H}_0+\delta\hat{H}, \tag{39}$$

where we denote the sum of atom and electromagnetic field Hamiltonian as $\hat{H}_0$ to indicate we treat them as unperturbed Hamiltonian while we will treat the interaction Hamiltonian $\delta\hat{H}$ as perturbation and time dependent perturbation theory will be applied. $\hat{H}_{\mathrm{EM}}$ has been solved in the previous section, and we assume the atom Hamiltonian has been solved too, so the eigenstate of $\hat{H}_0$ is tensor product of atom and electromagnetic field eigenstate. Let us put the detector at position $\mathbf{x}$, and consider the electric dipole interaction

$$\delta\hat{H}=-\hat{\mathbf{d}}\cdot\hat{\mathbf{E}}(\mathbf{x}), \tag{40}$$

where $\hat{\mathbf{d}}$ is the electric dipole operator of the atom, which is proportional to its position operator, and $\hat{\mathbf{E}}(\mathbf{x})$ is the electric field operator in Eq. (22). If there is no interaction term, atom and light live in different Hilbert space and mind their own business, nothing interesting happens. With the $\delta\hat{H}$ term, two systems talk to each other, and the ground state atom will be kicked to excited state accompanied by photon annihilation (We have seen this in 8.06x problem sets [7]). The calculation in this section will be more complicated than previous sections but the purpose is clear: *we want to calculate the probability we hear a click in our photodetector*. This corresponds to the transition probability where the atom stays in ground state $|g\rangle$ with the electromagnetic field in state $|\psi_i\rangle$ at time $t=0$, and after some time t the atom goes to excited state $|e\rangle$ while the electromagnetic field ends in state $|\psi_f\rangle$. The initial and final state of the system is

$$|i\rangle=|g\rangle\otimes|\psi_i\rangle\,, \tag{41}$$
$$|f\rangle=|e\rangle\otimes|\psi_f\rangle\,. \tag{42}$$

Go to the interaction picture where $\delta\hat{H}(t)=\exp(i\hat{H}_0t/\hbar)\delta\hat{H}\exp(-i\hat{H}_0t/\hbar)$. The time evolution operator will factorize into atom and field two parts, each acting on its corresponding operator in $\delta\hat{H}$

$$\delta\hat{H}(t) = -e^{i\hat{H}_{\mathrm{at}}t/\hbar}\hat{\mathbf{d}}e^{-i\hat{H}_{\mathrm{at}}t/\hbar}\cdot\hat{\mathbf{E}}(\mathbf{x},t), \tag{43}$$

with $\hat{\mathbf{E}}(\mathbf{x},t)$ the Heisenberg operator version of electromagnetic field operator in Eq. (29). According to first order time dependent perturbation theory [8], the transition amplitude of the event $|i\rangle \to |f\rangle$ at time t is

$$\begin{aligned}
&\frac{1}{i\hbar}\int_0^t dt' \langle f|\,\delta\hat{H}(t')\,|i\rangle = -\frac{1}{i\hbar}\int_0^t dt' \langle\psi_f|\,\hat{\mathbf{E}}(\mathbf{x},t')\,|\psi_i\rangle \\
&\cdot \langle e|\, e^{i\hat{H}_{\mathrm{at}}t'/\hbar}\hat{\mathbf{d}}e^{-i\hat{H}_{\mathrm{at}}t'/\hbar}\,|g\rangle \\
&= -\frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{eg}t'}\langle e|\,\hat{\mathbf{d}}\,|g\rangle\cdot\langle\psi_f|\,\hat{\mathbf{E}}(\mathbf{x},t')\,|\psi_i\rangle\,, \qquad (44)
\end{aligned}$$

with $\omega_{eg} = (E_e - E_g)/\hbar$, where E_e, E_g are energy of state $|e\rangle, |g\rangle$ respectively. Recall $\hat{\mathbf{E}}(\mathbf{x},t')$ can be break into positive and negative parts defined in Eq. (30) and Eq. (31), one with its terms proportional to $\hat{a}_m\exp(-i\omega_m t')$, the other $\hat{a}_m^\dagger\exp(i\omega_m t')$. Since ω_m and ω_{eg} are both positive, only terms proportional to $\exp(i(\omega_{eg}-\omega_m)t')$ with $\omega_{eg}-\omega_m \approx 0$ will contribute much to the integral(remember the stationary phase argument), while terms associated with negative parts are all proportional to $\exp(i(\omega_{eg}+\omega_m)t')$, oscillating fast in the integral. As a result, it is reasonable to keep only the positive part of the electric field operator when calculating the transition amplitude, so we have

$$\begin{aligned}
&\frac{1}{i\hbar}\int_0^t dt' \langle f|\,\delta\hat{H}(t')\,|i\rangle \\
&= -\frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{eg}t'}\langle e|\,\hat{\mathbf{d}}\,|g\rangle\cdot\langle\psi_f|\,\hat{\mathbf{E}}^{+}(\mathbf{x},t')\,|\psi_i\rangle\,. \qquad (45)
\end{aligned}$$

The modulus squared of the transition amplitude in Eq. (45) is the transition probability at time t

$$\begin{aligned}
P_{e,f\leftarrow g,i}(t) &= \left(\frac{1}{\hbar}\right)^2\sum_{\mu,\nu}\int_0^t dt'\int_0^t dt'' e^{i\omega_{eg}(t'-t'')} \\
&\times \langle g|\,\hat{d}_\nu\,|e\rangle\langle e|\,\hat{d}_\mu\,|g\rangle\langle\psi_i|\,\hat{E}_\nu^{-}(\mathbf{x},t'')\,|\psi_f\rangle\langle\psi_f|\,\hat{E}_\mu^{+}(\mathbf{x},t')\,|\psi_i\rangle\,, \qquad (46)
\end{aligned}$$

where we express the inner product of two vectors in its component form. For detectors that do not distinguish final state, we can sum over both atom excited state e and field final state f

$$\begin{aligned}
&P_{g,i}(t) = \sum_{e,f}P_{e,f\leftarrow g,i}(t) \\
&= \left(\frac{1}{\hbar}\right)^2\sum_{\mu,\nu,e}\int_0^t dt'\int_0^t dt'' e^{i\omega_{eg}(t'-t'')} \\
&\times \langle g|\,\hat{d}_\nu\,|e\rangle\langle e|\,\hat{d}_\mu\,|g\rangle\langle\psi_i|\,\hat{E}_\nu^{-}(\mathbf{x},t'')\hat{E}_\mu^{+}(\mathbf{x},t')\,|\psi_i\rangle\,, \qquad (47)
\end{aligned}$$

where we use the complete relation: $\sum_f |\psi_f\rangle\langle\psi_f| = \mathbb{1}$. One more approximation is to replace sum over e with integral, $\sum_e = \int d\omega_e n(\omega_e)$, where $n(\omega_e)$ is the usual density of state. To make the expression less messy, let us make a few more definitions. Group everything associated with electric field operators as

$$G_{\nu\mu}(\mathbf{x},t'';\mathbf{x},t') = \langle\psi_i|\,\hat{E}_\nu^{-}(\mathbf{x},t'')\hat{E}_\mu^{+}(\mathbf{x},t')\,|\psi_i\rangle\,, \tag{48}$$

which is called correlation function. Group everything about the atom along with the overall constant as

$$\begin{aligned}
s_{\nu\mu}(t'-t'') &= \hbar^{-2}\int d\omega_e n(\omega_e)\langle g|\,\hat{d}_\nu\,|e\rangle\langle e|\,\hat{d}_\mu\,|g\rangle e^{i\omega_{eg}(t'-t'')} \\
&= \int d\omega_e s_{\nu\mu}(\omega_e)e^{i\omega_{eg}(t'-t'')}, \qquad (49)
\end{aligned}$$

where $s_{\nu\mu}(\omega_e) = \hbar^{-2}n(\omega_e)\langle g|\,\hat{d}_\nu\,|e\rangle\langle e|\,\hat{d}_\mu\,|g\rangle$ is called the frequency response function and $s_{\nu\mu}(t'-t'')$ sensitivity function. The transition probability in Eq. (47) can be expressed as

$$\begin{aligned}
P_{g,i}(t) &= \sum_{\mu,\nu}\int_0^t dt'\int d\omega_e s_{\nu\mu}(\omega_e) \\
&\times \int_0^t dt'' e^{i\omega_{eg}(t'-t'')}G_{\nu\mu}(\mathbf{x},t'';\mathbf{x},t'). \qquad (50)
\end{aligned}$$

The integral in second line of Eq. (50) becomes negligible if $\omega_{eg} = \omega_e - \omega_g \gg 1$ since the integrand oscillates too fast, so $s_{\nu\mu}(\omega_e)$ only contributes in some narrow band. Let $s_{\nu\mu}(\omega_e) = s_{\nu\mu}$ in this narrow band and pull it out from the integral. Notice $\int d\omega_e e^{i\omega_{eg}(t'-t'')} = \delta(t'-t'')$ (We can extend the lower limit to minus infinity because its contribution in the original integral is negligible.), so the final result is

$$P_{g,i}(t) = \sum_{\mu,\nu}\int_0^t dt' s_{\nu\mu}G_{\nu\mu}(\mathbf{x},t';\mathbf{x},t'). \tag{51}$$

The detection rate is

$$\begin{aligned}
R(t) &= \frac{dP_{g,i}(t)}{dt} = \sum_{\mu,\nu}s_{\nu\mu}G_{\nu\mu}(\mathbf{x},t;\mathbf{x},t) \\
&= \sum_{\mu,\nu}s_{\nu\mu}\langle\psi_i|\,\hat{E}_\nu^{-}(\mathbf{x},t)\hat{E}_\mu^{+}(\mathbf{x},t)\,|\psi_i\rangle\,. \qquad (52)
\end{aligned}$$

Notice further that $s_{\nu\mu}$ involves the moment of atom position operators because $\hat{d}_\nu\hat{d}_\mu \propto \hat{x}_\nu\hat{x}_\mu$, so for isotropic detectors, we may have $s_{\nu\mu} = s\delta_{\nu\mu}$ for some constant number s. For such detectors, the detection rate is

$$R(t) = s\,\langle\psi_i|\,\hat{\mathbf{E}}^{-}(\mathbf{x},t)\cdot\hat{\mathbf{E}}^{+}(\mathbf{x},t)\,|\psi_i\rangle\,. \tag{53}$$

As promised before, the effect of electric field enters the detection rate of photoelectric detector as $\langle\psi_i|\,\hat{\mathbf{E}}^{-}(\mathbf{x},t)\cdot\hat{\mathbf{E}}^{+}(\mathbf{x},t)\,|\psi_i\rangle$.

B. No Double Detection

What is the detection rate for a detector placed at position $\mathbf{x}$ at time t if we have initial field state as one-photon multimode state $|\psi_i\rangle = |1\rangle$? We actually have calculated the moment $\langle 1| \hat{\mathbf{E}}^{-}(\mathbf{x},t) \cdot \hat{\mathbf{E}}^{+}(\mathbf{x},t) |1\rangle$ in Eq. (36), copy the result and plug it in Eq. (53)

$$R(\mathbf{x},t) = s\sum_{m,n} 2\pi\hbar\sqrt{\omega_m\omega_n}(\mathbf{u}_m \cdot \mathbf{u}_n)c_m^* c_n e^{i(\omega_m-\omega_n)t}. \tag{54}$$

In the single-photon multimode state, $R(\mathbf{x},t)$ can be interpreted as the probability of finding the photon at position $\mathbf{x}$ at time t and it is analogous to the modulus squared of the wavefunction of a particle we learned in quantum mechanics. However, nothing in principle can distinguish $|1\rangle$ from a classical field with the electric field amplitude specified by right choice of $c_m, \mathbf{u}_m$ and ω_m so that the detection rate is identical to what we have in Eq. (54).

What if we put two detectors at two different places and try to detect light at different positions $\mathbf{x}_1, \mathbf{x}_2$ at the same time? Classical, the probability of hearing two clicks at the same time is the product of rate $R(\mathbf{x}_1,t)R(\mathbf{x}_2,t)$. However, if we are in state $|1\rangle$, the probability to detect the photon at different places at the same time should be 0. We just have *one* photon in spacetime! Let us now demonstrate this. There are two detectors, so the interaction Hamiltonian in Eq. (39) should be

$$\delta\hat{H}_2 = -\hat{\mathbf{d}}_1 \cdot \hat{\mathbf{E}}(\mathbf{x}_1) - \hat{\mathbf{d}}_2 \cdot \hat{\mathbf{E}}(\mathbf{x}_2). \tag{55}$$

In the interaction picture

$$\begin{aligned}\delta\hat{H}_2(t) = &-e^{i\hat{H}_{\mathrm{at1}}t/\hbar}\hat{\mathbf{d}}_1 e^{-i\hat{H}_{\mathrm{at1}}t/\hbar} \cdot \hat{\mathbf{E}}(\mathbf{x}_1,t)\\ &- e^{i\hat{H}_{\mathrm{at2}}t/\hbar}\hat{\mathbf{d}}_2 e^{-i\hat{H}_{\mathrm{at2}}t/\hbar} \cdot \hat{\mathbf{E}}(\mathbf{x}_2,t),\end{aligned} \tag{56}$$

where $\hat{H}_{\mathrm{at1}}, \hat{H}_{\mathrm{at2}}$ are Hamiltonian of atom at position $\mathbf{x}_1, \mathbf{x}_2$ respectively. The initial and final state is

$$|i\rangle = |g_1\rangle \otimes |g_2\rangle \otimes |\psi_i\rangle\,, \tag{57}$$
$$|f\rangle = |e_1\rangle \otimes |e_2\rangle \otimes |\psi_f\rangle\,. \tag{58}$$

Since $\langle f| \delta\hat{H}_2(t) |i\rangle = 0$, first order correction vanishes. We must calculate the second order correction. It will not be surprising to you that the second order correction result for the transition rate is

$$\begin{aligned}&R(\mathbf{x}_1,t;\mathbf{x}_2,t) = s^2\\ &\times \sum_{l,m} \langle\psi_i| \hat{E}_l^{-}(\mathbf{x}_1,t)\hat{E}_m^{-}(\mathbf{x}_2,t)\hat{E}_m^{+}(\mathbf{x}_2,t)\hat{E}_l^{+}(\mathbf{x}_1,t) |\psi_i\rangle\,.\end{aligned} \tag{59}$$

The detailed derivation can be found in Cohen-Tannoudji's book [6]. Let $|\psi_i\rangle = |1\rangle$, the detection rate of a single photon at two different places at the same time indeed vanishes

$$\begin{aligned}&R(\mathbf{x}_1,t;\mathbf{x}_2,t) = s^2\\ &\times \sum_{l,m} \langle 1| \hat{E}_l^{-}(\mathbf{x}_1,t)\hat{E}_m^{-}(\mathbf{x}_2,t)\hat{E}_m^{+}(\mathbf{x}_2,t)\hat{E}_l^{+}(\mathbf{x}_1,t) |1\rangle\,.\\ &= 0,\end{aligned} \tag{60}$$

since $|1\rangle$ can not survive two consecutive annihilation operators. This unique property of single-photon multimode state was shown experimentally in 1986 by Grangier, Roger and Aspect [9].

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